

Inverse scattering with non-overdetermined data

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Abstract

Let $A(\beta, \alpha, k)$ be the scattering amplitude corresponding to a real-valued potential which vanishes outside of a bounded domain $D \subset \mathbb{R}^3$. The unit vector α is the direction of the incident plane wave, the unit vector β is the direction of the scattered wave, $k > 0$ is the wave number. The governing equation for the waves is $[\nabla^2 + k^2 - q(x)]u = 0$ in \mathbb{R}^3 .

For a suitable class of potentials it is proved that if $A_{q_1}(-\beta, \beta, k) = A_{q_2}(-\beta, \beta, k) \forall \beta \in S^2, \forall k \in (k_0, k_1)$, and $q_1, q_2 \in M$, then $q_1 = q_2$. This is a uniqueness theorem for the solution to the inverse scattering problem with backscattering data.

It is also proved for this class of potentials that if $A_{q_1}(\beta, \alpha_0, k) = A_{q_2}(\beta, \alpha_0, k) \forall \beta \in S_1^2, \forall k \in (k_0, k_1)$, and $q_1, q_2 \in M$, then $q_1 = q_2$.

Here S_1^2 is an arbitrarily small open subset of S^2 , and $|k_0 - k_1| > 0$ is arbitrarily small.

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1 Introduction

Consider the scattering problem:

$$Lu := [\nabla^2 + k^2 - q(x)]u = 0 \quad \text{in } \mathbb{R}^3, \quad k = \text{const} > 0, \quad (1)$$

$$u = e^{ik\alpha \cdot x} + A(\beta, \alpha, k) \frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r := |x| \rightarrow \infty, \quad \beta = \frac{x}{r}, \quad \alpha \in S^2, \quad (2)$$

where S^2 is the unit sphere in \mathbb{R}^3 , and $A(\beta, \alpha, k) = A_q(\beta, \alpha, k)$ is the scattering amplitude corresponding to the potential $q(x)$, α is the direction of the incident plane wave, β is a direction of the scattered wave, and k^2 is the energy.

Let us assume that q is a real-valued compactly supported function,

$$q \in M := W_0^{\ell,1}(D), \quad \ell > 2,$$

$D \subset \mathbb{R}^3$ is a bounded domain, and $W_0^{\ell,1}(D)$ is the Sobolev space, it is the closure of $C_0^\infty(D)$ in the norm of the Sobolev space $W^{\ell,1}(D)$. This space consists of the functions whose derivatives up to the order ℓ are absolutely integrable in D .

The inverse scattering problems, we are studying in this paper, are:

IP1: Do the backscattering data $A(-\beta, \beta, k)$ known $\forall k > 0, \forall \beta \in S^2$, determine $q \in M$ uniquely?

IP2: Do the data $A_q(\beta, k) := A(\beta, \alpha_0, k)$ known $\forall k > 0, \forall \beta \in S^2$, determine $q \in M$ uniquely?

We give a positive answer to these questions. Theorem 1 (see below) is our basic result.

These inverse problems have been open for many decades (see, e.g., [7]). They are a part of the general question in physics: does the S -matrix determine the Hamiltonian uniquely?

It was known that the data $A(\beta, \alpha, k) \forall \alpha, \beta \in S^2, \forall k > 0$, determine $q(x) \in C^1(\mathbb{R}^3) \cap C(\mathbb{R}^3, (1+|x|)^\gamma, \gamma > 3)$ uniquely. Here $\|q\|_{C(\mathbb{R}^3, (1+|x|)^\gamma)} = \sup_{x \in \mathbb{R}^3} \{(1+|x|)^\gamma |q(x)|\}$, and the datum $A(\beta, \alpha, k)$ is a function of 5 variables (two unit vectors $\beta, \alpha \in S^2$ and a scalar $k > 0$), while the potential q is a function of 3 variables, (x_1, x_2, x_3) . We are not stating this old result with minimal assumptions on the class of potentials.

The author proved (see [2]-[7]) that the data $A_q(\beta, \alpha) := A_q(\beta, \alpha, k)$, known $\forall \alpha \in S_1^2, \forall \beta \in S_2^2$ and a fixed $k = k_0 > 0$, determine $q \in Q_a$ uniquely. Here $S_j^2, j = 1, 2$, are arbitrary small open subsets of S^2 (solid angles), and

$$Q_a := \{q : q = \bar{q}, q = 0 \text{ if } |x| > a, \quad q \in L^2(B_a)\}, \quad B_a := \{x : |x| \leq a\},$$

$a > 0$ is an arbitrary large fixed number. In this uniqueness theorem the datum $A_q(\beta, \alpha)$ is a function of four variables (two unit vectors $\alpha, \beta \in S^2$) and the potential q is a function of three variables (x_1, x_2, x_3) . Therefore, this inverse problem is also overdetermined.

It is natural to assume that q has compact support in a study of the inverse scattering problem, because in practice the data are always noisy, and from noisy data it is *in principle impossible* to determine the rate of decay of a potential $q(x)$, such that $|q(x)| \leq c(1+|x|)^{-\gamma}, \gamma > 3$, for all sufficiently large $|x|$. Indeed, the contribution of the "tail" of q , that is, of the function $q_R := q_R(x)$,

$$q_R(x) := \begin{cases} 0, & |x| \leq R, \\ q(x), & |x| > R, \end{cases}$$

to the scattering amplitude cannot be distinguished from the contribution of the noise if R is sufficiently large. For example, if the noisy data are $A_q^{(\delta)}(\beta, \alpha, k)$,

$$\sup_{\beta, \alpha \in S^2} |A_q^{(\delta)}(\beta, \alpha, k) - A_q(\beta, \alpha, k)| < \delta,$$

then one can prove that the contribution of q_R to A_q is $O(\frac{1}{R^{\gamma-3}})$. Thus, this contribution is of the order of the noise level δ if $R = O(\delta^{1/(3-\gamma)})$, $\gamma > 3$. This yields an estimate of the "radius of compactness" of the potential q given the

noise level δ and the exponent $\gamma > 3$, which describes the rate of decay of the potential.

There were no results concerning the uniqueness of the solution to the inverse scattering problems IP1 and IP2 with the non-overdetermined backscattering data $A(-\beta, \beta, k) \forall \beta \in S^2, \forall k > 0$, or with the non-overdetermined data $A(\beta, \alpha_0, k) \forall \beta \in S^2, \forall k > 0, \alpha = \alpha_0$ being fixed.

The main result of this paper is:

Theorem 1. 1) If $A_{q_1}(-\beta, \beta, k) = A_{q_2}(-\beta, \beta, k) \forall \beta \in S^2, \forall k > 0$ and $q_j \in M, j = 1, 2$, then $q_1 = q_2$.

2) If $A_{q_1}(\beta, \alpha_0, k) = A_{q_2}(\beta, \alpha_0, k) \forall \beta \in S^2, \forall k > 0, \alpha_0 \in S^2$ is fixed, and $q_j \in M, j = 1, 2$, then $q_1 = q_2$.

Remark 1. Theorem 1 remains valid if the data are given $\forall \beta \in S_1^2, \forall k \in (k_0, k_1), 0 < k_0 < k_1$, where S^2 and $|k_1 - k_0| > 0$ is arbitrarily small.

Indeed, if $q \in M$, or, more generally, if q is compactly supported, $\text{supp } q \subset B_a$, and $q \in L^2(B_a)$, then the author has proved (see [7] and [8]), that $A(\beta, \alpha, k)$ is a restriction to $(0, \infty)$ of a meromorphic in \mathbb{C} function of k and a restriction to $S^2 \times S^2$ of a function analytic on the variety $\mathcal{M} \times \mathcal{M}, \mathcal{M} := \{\theta : \theta \in \mathbb{C}^3, \theta \cdot \theta = 1\}$, where $\theta \cdot \theta := \sum_{j=1}^3 \theta_j^2$. Therefore, if $A(\beta, \alpha_0, k)$ is known on $S_1^2 \times (k_0, k_1)$ then it is uniquely determined on $S^2 \times (0, \infty)$ by analytic continuation.

The algebraic variety \mathcal{M} is a non-compact algebraic variety in \mathbb{C}^3 .

Remark 2. The main idea of the proof of Theorem 1 is to establish completeness of the set of products of the scattering solutions in a class M of potentials. This is a version of Property C, introduced and applied by the author to many inverse problems (see [3], [5], [6], [7]).

2 Proofs

The following lemma is crucial for the proof of both statements of Theorem 1.

Lemma 1. ([7, p.262]) If $p(x) := q_1(x) - q_2(x)$, then

$$-4\pi[A_{q_1}(\beta, \alpha, k) - A_{q_2}(\beta, \alpha, k)] = \int_D p(x)u_1(x, \alpha, k)u_2(x, -\beta, k)dx. \quad (3)$$

In (3) u_j are the scattering solutions, that is, solutions to (1)-(2) with $q = q_j$, or, equivalently, solutions to the integral equation:

$$u_j(x, \alpha, k) = e^{ik\alpha \cdot x} - \int_D g(x, y, k)q_j(y)u_j(y, \alpha, k)dy, \quad g(x, y, k) := \frac{e^{ik|x-y|}}{4\pi|x-y|}. \quad (4)$$

Let $v_j := e^{-ik\alpha \cdot x}u_j$. Then

$$u_j = e^{ik\alpha \cdot x}[1 + \epsilon_j], \quad \epsilon_j := - \int_D G(x, y, k)q_j(y)v_j(y, \alpha, k)dy, \quad (5)$$

where

$$G(x, y, k) := g(x, y, k)e^{-ik\alpha \cdot (x-y)}.$$

The function v_j solves the integral equation

$$v_j = 1 - B_j v_j, \quad B_j v_j := - \int_D G(x, y, k) q_j(y) v_j(y, \alpha, k) dy, \quad (6)$$

and $B_j v_j = \epsilon_j$.

If $A_{q_1} = A_{q_2} \forall \beta \in S^2, \forall k > 0$, and $\beta = -\alpha$, then (3) yields the following *orthogonality relation*:

$$\int_D p(x) u_1(x, \beta, k) u_2(x, \beta, k) dx = 0, \quad \forall \beta \in S^2, \quad \forall k > 0, \quad (7)$$

where

$$p(x) = q_1(x) - q_2(x).$$

The IP2 is treated similarly.

The orthogonality relation (7) can be written as

$$\int_D p(x) e^{2ik\beta \cdot x} [1 + \epsilon(x, \beta, k)] dx = 0, \quad \forall \beta \in S^2, \quad \forall k > 0, \quad \epsilon := \epsilon_1 + \epsilon_2 + \epsilon_1 \epsilon_2. \quad (8)$$

The relation (8) holds for $\Im k \geq 0, k \neq i\kappa_{m,j}$, where $i\kappa_{m,j}, 1 \leq m \leq m_j, j = 1, 2$, are the numbers at which the operator $I + B_j$ is not injective. There are finitely many such numbers in the upper half complex plane if $q_j \in M$. The numbers $\kappa_{m,j} > 0, -\kappa_{m,j}^2$ are the negative eigenvalues of the Schroedinger operator L_j in $L^2(\mathbb{R}^3)$, where L_j is the operator in (1) with $q = q_j$.

In what follows we write ϵ meaning ϵ_j for $j = 1, 2$, or ϵ , defined in (8). Also, we write κ_m in place of $\kappa_{m,j}$. This will not cause any confusion.

Since q is compactly supported, the scattering solution $u(x, \alpha, k)$ is analytic in the region $\Im k \geq 0$, except, possibly, for a finite number of poles $k_m = i\kappa_m, \kappa_m > 0, \kappa_m < \kappa_{m+1}, 1 \leq m \leq m_0 < \infty$, where $m_0 < \infty$ is a positive integer. Therefore, $u(x, \alpha, k)$ and $\epsilon(x, \alpha, k)$ are analytic in the region $\Im k \geq 0, k \neq k_m, 1 \leq m \leq m_0$. Let $\eta_0 > 0$ be chosen so that $\eta_0 > \max_m \kappa_m$.

The orthogonality relation (8) for $q_j \in M$ holds in the region $\Im k \geq 0, k \neq i\kappa_m$, and the integrand in (8) is analytic with respect to k in this region.

We want to derive from (8) that $p(x) = 0$.

Write the orthogonality relation (8) as:

$$\tilde{p}(2k\beta) + (2\pi)^{-3} \tilde{p} \star \tilde{\epsilon} = 0, \quad (9)$$

where the \star denotes convolution,

$$\tilde{p}(\xi) := \int_{\mathbb{R}^3} e^{i\xi \cdot x} p(x) dx, \quad \tilde{p} \star \tilde{\epsilon} := \int_{\mathbb{R}^3} \tilde{p}(\xi - \nu) \tilde{\epsilon}(\nu) d\nu, \quad (10)$$

and in (9) $\tilde{p} \star \tilde{\epsilon}$ is calculated at $\xi = 2k\beta$.

Equation (9) has only the trivial solution $\tilde{p} = 0$ provided that

$$(2\pi)^{-3} \|\tilde{\epsilon}(\xi, \beta, k)\|_1 < b < 1, \quad (11)$$

where

$$\|\tilde{\epsilon}\|_1 = \int_{\mathbb{R}^3} |\tilde{\epsilon}(\xi, \beta, k)| d\xi.$$

Indeed,

$$\max_{k \geq 0, \beta \in S^2} |\tilde{p}(2k\beta)| \leq \max_{k \geq 0, \beta \in S^2, \nu \in \mathbb{R}^3} |\tilde{p}(2k\beta - \nu)| \cdot \|\tilde{\epsilon}\|_1 < \max_{k \geq 0, \beta \in S^2} |\tilde{p}(2k\beta)|, \quad (12)$$

where we have taken into account that the sets

$$\{2k\beta\}_{\forall k \geq 0, \forall \beta \in S^2}$$

and

$$\{2k\beta - \nu\}_{\forall k \geq 0, \forall \beta \in S^2, \forall \nu \in \mathbb{R}^3}$$

are the same.

Inequalities (11) and (12) imply

$$\tilde{p}(2k\beta) = 0 \quad \forall k > 0, \forall \beta \in S^2.$$

If $\tilde{p}(2k\beta) = 0 \quad \forall k > 0, \forall \beta \in S^2$, then $\tilde{p} = 0$, and, by the injectivity of the Fourier transform, one concludes that $p = 0$.

Since p is compactly supported, the function \tilde{p} is entire function of ξ . Consequently, if one proves that $\tilde{p}(2(k + i\eta)\beta) = 0 \quad \forall k > 0, \forall \beta \in S^2$, and for $\eta > \eta_0 > 0$, then $\tilde{p} = 0$ by analytic continuation, and, consequently, $p = 0$. This observation is used below.

Thus, to prove the first claim of Theorem 1, it is sufficient to establish inequality (11).

However, (11) with $k > 0$ does not hold because the function $\frac{1}{|\xi|^2 - 2k\beta \cdot \xi}$ (see formula (16) below) is not absolutely integrable if $k > 0$.

The idea, that makes the proof work, is to replace $k > 0$ with $k + i\eta$, where $\eta > \eta_0 > 0$ is sufficiently large. The orthogonality relation (7) remains valid after such a replacement because of the analyticity of $\epsilon = \epsilon(x, \beta, k)$ with respect to k in the region $\Im k > \eta_0$. Equation (8) holds with $k + i\eta$ replacing k .

The argument, given in (12), remains valid after this replacement because

$$\mu := \max_{k > 0, \eta \in (\eta_0, \eta_1), \beta \in S^2} |\tilde{p}(2(k + i\eta)\beta)| \geq c \max_{\xi \in \mathbb{R}^3} |\tilde{p}(\xi)| := c\mu_1,$$

where $c > 0$ is a constant and $\eta_1 > \eta_0$ is a sufficiently large number, which is assumed finite in order to have $\mu < \infty$.

Therefore, (9) with $k + i\eta$ replacing k yields:

$$\mu \leq \max_{k > 0, \eta \in (\eta_0, \eta_1), \beta \in S^2} \int_{\mathbb{R}^3} |\tilde{\epsilon}(2(k + i\eta)\beta - \xi)| d\xi \quad \mu_1 < \mu,$$

and, consequently, $\mu = 0$ and $p(x) = 0$, provided that an analog of (11) holds:

$$\max_{k>0, \eta \in (\eta_0, \eta_1), \beta \in S^2} \int_{\mathbb{R}^3} |\tilde{e}(2(k + i\eta)\beta - \xi)| d\xi < b(\eta),$$

where

$$\lim_{\eta \rightarrow +\infty} b(\eta) = 0,$$

so that

$$cb(\eta) < 1, \quad \eta > \eta_0,$$

for sufficiently large $\eta > \eta_0$.

We refer to this inequality also as (11), and prove that this inequality holds if η is sufficiently large (see (18) below, from which it follows that

$$b(\eta) = O(|\eta|^{-1}) \quad \eta \rightarrow +\infty.$$

Let us check that

$$\mu \geq c\mu_1.$$

This inequality will be established if one proves that

$$\mu = \sup_{\beta \in S^2, k>0, \eta \in (\eta_0, \eta_1)} |\tilde{p}((k + i\eta)\beta)| \geq c \int_D |p(x)| dx,$$

because

$$\sup_{\xi \in \mathbb{R}^3} |\tilde{p}(\xi)| \leq \int_D |p(x)| dx.$$

One has

$$\mu \geq \sup_{\beta \in S^2, \eta \in (\eta_0, \eta_1)} \left| \int_D e^{-2\eta\beta \cdot x} p(x) dx \right| = \sup_{\beta \in S^2, \eta \in (\eta_0, \eta_1)} |W|,$$

where

$$W := \int_D e^{-2\eta\beta \cdot x} p(x) dx.$$

Let us prove that

$$\sup_{\beta \in S^2, \eta \in (\eta_0, \eta_1)} |W| \geq c \int_D |p(x)| dx.$$

If this inequality is established, then the proof of the inequality $\mu \geq c\mu_1$ is complete.

We may assume that $p \not\equiv 0$, because otherwise there is nothing to prove. If $p \not\equiv 0$, then $W \not\equiv 0$. The function W is an entire function of the vector $\eta\beta$, considered as a vector in \mathbb{C}^3 . The function $\sup_{\beta \in S^2} |W|$ tends to ∞ as $\eta \rightarrow +\infty$ (see [1] for the growth rates of entire functions of exponential type). Therefore inequality $\sup_{\beta \in S^2, \eta \in (\eta_0, \eta_1)} |W| \geq c \int_D |p(x)| dx$ holds, and inequality $\mu \geq c\mu_1$ is established.

If inequality (11) is proved for $k + i\eta$ replacing k , then the argument, similar to the one, given in (12), yields $\tilde{p}(2(k + i\eta)\beta) = 0$ for all $k > 0$, $\beta \in S^2$, and $\eta > \eta_0$. By the analytic continuation this implies $\tilde{p}(\xi) = 0$ for all ξ , so $p(x) = 0$.

The first claim of Theorem 1 is therefore proved as soon as estimate (11) is proved with $k + i\eta$ replacing k .

Let us now establish inequality (11) with $k + i\eta$ replacing k .

Note that

$$\epsilon = - \int_D \frac{e^{ik[|x-y|-\beta \cdot (x-y)]}}{4\pi|x-y|} \psi(y) dy, \quad \psi := qv.$$

Using the Fourier transform of convolution, one gets

$$\tilde{\epsilon} = -F\left(\frac{e^{ik[|x|-\beta \cdot x]}}{4\pi|x|}\right)F(qv), \quad F(\psi) := \tilde{\psi}. \quad (13)$$

The assumption $q \in W_0^{\ell,1}(D)$ and the elliptic regularity results for v , which solves a second-order elliptic equation, imply that v is smoother than q , and, therefore, $\psi = qv$ belongs to $W_0^{\ell,1}(D)$, $\psi \in W_0^{\ell,1}(\mathbb{R}^3)$, $\ell > 2$.

Let us now derive the estimate (14), given below.

If a function $f \in L^1(\mathbb{R}^3)$, then $|f| \leq c$. Here and below by $c > 0$ we denote various constants.

If $f \in W_0^{\ell,1}(D)$, then $D^\ell f \in L^1(\mathbb{R}^3)$, where D^ℓ stands for any derivative of order ℓ . Therefore $|F(D^\ell f)| = |\xi^\ell \tilde{f}| \leq c$. If f is compactly supported, then $\tilde{f} \in C_{loc}^\infty(\mathbb{R}^3)$, and the estimate $|\xi^\ell \tilde{f}| \leq c$ implies the inequality

$$\sup_{\xi \in \mathbb{R}^3} (1 + |\xi|)^\ell |\tilde{f}| < c.$$

We apply this inequality to the function $f = qv := \psi \in W_0^{\ell,1}(D)$ and get:

$$(1 + |\xi|)^\ell |\tilde{\psi}| < c, \quad \ell > 2. \quad (14)$$

Let us calculate now the first factor on the right-hand side of equation (13). We have

$$\int_{\mathbb{R}^3} e^{i\xi \cdot x} \frac{e^{ik[|x|-\beta \cdot x]}}{4\pi|x|} = -\frac{1}{|\xi|^2 - 2k\beta \cdot \xi}. \quad (15)$$

Therefore

$$\tilde{\epsilon} = -\frac{\tilde{\psi}(\xi)}{|\xi|^2 - 2k\beta \cdot \xi}. \quad (16)$$

Let us replace k by $k + i\eta$ in (15) and (16). In $\tilde{\psi}$ the dependence on k enters through v . Choose $\eta > \eta_0 > 0$ sufficiently large, so that the integral I in (18) (see below) will be as small as we wish. This will yield estimate (11) with $k + i\eta$ replacing k .

Using the spherical coordinates with the z -axis directed along β , $t = \cos \theta$, θ is the angle between β and $x - y$, $r := |x - y|$, and using estimate (14), one gets:

$$\|\tilde{\epsilon}\|_1 \leq c \int_0^\infty \frac{dr r}{(1+r)^\ell} \int_{-1}^1 \frac{dt}{[r - 2kt]^2 + 4\eta^2 t^2]^{1/2}} := cI. \quad (17)$$

The integral with respect to t in (17) can be calculated in closed form, and one gets:

$$I = \frac{1}{2(k^2 + \eta^2)^{1/2}} \int_0^\infty \frac{dr}{(1+r)^\ell} \log \left| \frac{1-a+[(1-a)^2+b]^{1/2}}{-1-a+[(1+a)^2+b]^{1/2}} \right|, \quad (18)$$

where

$$a := \frac{kr}{2(k^2 + \eta^2)}, \quad b := \frac{\eta^2 r^2}{4(k^2 + \eta^2)}. \quad (19)$$

If $r \rightarrow \infty$, then the ratio under the log sign in (18) tends to 1, and, since $\ell > 2$, the integral in (18) converges.

If $\eta > 0$ is sufficiently large, then estimate (18) implies that the inequality (11) holds with k replaced by $k+i\eta$. Therefore $\tilde{p}(2(k+i\eta)\beta) = 0 \ \forall k > 0, \forall \beta \in S^2$ and $\eta > \eta_0$. This implies $\tilde{p} = 0$, so $p = 0$, and the first claim of Theorem 1 is proved.

The second claim of Theorem 1 is proved similarly. One starts with the orthogonality relation

$$\int_D p(x) u_1(x, \alpha_0, k) u_2(x, \beta, k) dx = 0 \quad \forall k > 0, \forall \beta \in S^2,$$

writes it as

$$\int_D p(x) e^{ik(\alpha_0 + \beta) \cdot x} [1 + \epsilon] dx = 0 \quad \forall k > 0, \forall \beta \in S^2,$$

and, replacing k with $k + i\eta$, gets

$$\tilde{p}((k + i\eta)(\alpha_0 + \beta)) + (2\pi)^{-3} \tilde{p} \star \tilde{\epsilon} = 0.$$

Using estimate (11) with $k + i\eta$ replacing k , one obtains the relation

$$\tilde{p}((k + i\eta)(\alpha_0 + \beta)) = 0 \quad \forall k > 0, \forall \beta \in S^2, \quad \eta > \eta_0.$$

Since $\tilde{p}(\xi)$ is an entire function of $\xi \in \mathbb{C}^3$, this implies $\tilde{p} = 0$, so $p = 0$, and the second claim of Theorem 1 is proved.

Theorem 1 is proved □

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